LAMINAR BOUNDARY LAYER WITH UNIFORM SUCTION ON FLAT PLATE IN OSCILLATING FLOW

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We examine unsteady incompressible fluid flow in a laminar boundary layer with uniform suction for longitudinal flow over a flat plate when the external stream is a flow with constant velocity, on which there is superposed a sinusoidal disturbance convected by the stream, analogous to [1]. We study the stability of such flow in the boundary layer.

1. Velocity Field in Boundary Layer in the Presence of Periodic Disturbances in the Outer Stream. We assume that the external stream velocity has the form

$$U(x, t) = U_0[1 + \lambda \cos \omega(x / U_0 - t)] \tag{1.1}$$

and fluid suction with the constant velocity $v_0 < 0$ is provided along the entire wetted surface of the plate.

The equations of the unsteady two-dimensional boundary layer have the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} \tag{1.2}$$

The boundary conditions are

$$u(x, y, t) = 0, \quad v(x, y, t) = v_0 = \text{const} \quad (y = 0)$$
$$u(x, y, t) \to U(x, t) \quad (y \to \infty)$$

Here u(x,y,t) and v(x,y,t) are respectively the longitudinal and transverse velocity components in the boundary layer, p the pressure, ν the kinematic viscosity, the x axis is directed along the plate, the y axis is perpendicular to the plate.

In (1.2) we convert to the dimensionless variables

$$u = U_0 u^\circ, \; v = |v_0| v^\circ, \; \zeta = \frac{v_0^2 x}{U_0 \nu}, \; \eta = \frac{|v_0| y}{\nu}, \; \tau = \omega\left(\frac{x}{U_0} - t\right) \tag{1.3}$$

We seek those solutions for u° and v° which will be functions only of η and τ. These solutions will be applicable beginning only at some distance from the leading edge of the plate. Such solutions must satisfy the following equations

$$\frac{\partial^2 u^\circ}{\partial \eta^2} + \gamma(1 - u^0)\frac{\partial u^\circ}{\partial \tau} - v^\circ\frac{\partial u^\circ}{\partial \eta} = \frac{\lambda^2}{2}\gamma \sin 2\tau, \; \gamma\frac{\partial u^\circ}{\partial \tau} + \frac{\partial v^\circ}{\partial \eta} = 0 \tag{1.4}$$

while the boundary conditions

$$u^\circ = 0, \; v^\circ = -1 \quad (\eta = 0), \quad u^\circ \to 1 + \lambda\cos\tau \quad (\eta \to \infty), \quad \gamma = \nu\omega / v_0^2$$

Assuming $\lambda \ll 1$, we seek the solution of (1.4) in the form

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$$u^\circ(\eta, \tau) = u_0(\eta) + \lambda u_1(\eta,\tau) + \lambda^2 u_2(\eta,\tau) + \ldots$$
$$v^\circ(\eta, \tau) = w_0(\eta) + \lambda w_1(\eta,\tau) + \lambda^2 w_2(\eta, \tau) + \ldots \qquad (1.5)$$

Substituting (1.5) into (1.4) and collecting terms with the same powers of λ, we obtain the systems of equations for the coefficients of (1.5).

The terms with zero power of λ yield the stationary equations

$$w_0 \frac{\partial u_0}{\partial \eta} = \frac{\partial^2 u_0}{\partial \eta^2}, \qquad \frac{\partial w_0}{\partial \eta} = 0 \qquad (1.6)$$

with the boundary conditions

$$u_0 = 0, \quad w_0 = -1 \quad (\eta = 0), \quad u_0 \to 1 \ (\eta \to \infty)$$

These are known equations for the asymptotic suction profile on a flat plate. Their solution has the form

$$u_0 = 1 - e^{-\eta}, \qquad w_0 = -1 \qquad (1.7)$$

The terms with λ^1 yield the equations

$$L(u_1, w_1) = 0, \qquad K(u_1, w_1) = 0 \qquad (1.8)$$

with the boundary conditions

$$u_1 = w_1 = 0 \quad (\eta = 0), \quad u_1 \to \cos\tau \quad (\eta \to \infty)$$

The terms with λ^2 yield the equations

$$L(u_2, w_2) = \frac{\gamma}{2}\sin 2\tau + \gamma u_1 \frac{\partial u_1}{\partial \tau} + w_1 \frac{\partial u_1}{\partial \eta}$$
$$K(u_2, w_2) = 0 \qquad (1.9)$$

with the boundary conditions

$$u_2 = w_2 = 0 \quad (\eta = 0), \quad u_2 \to 0 \quad (\eta \to \infty)$$

The terms with λ^2 (k > 2) yield the equations

$$L(u_k, w_k) = \gamma \sum_{i=1}^{k-1} u_i \frac{\partial u_{k-i}}{\partial \tau} + \sum_{i=1}^{k-1} w_i \frac{\partial u_{k-i}}{\partial \eta}$$
$$K(u_k, w_k) = 0 \qquad (1.10)$$

with the boundary conditions

$$u_k = w_k = 0 \quad (\eta = 0), \quad u_k \to 0 \quad (\eta \to \infty)$$

Here

$$L(u, w) = \frac{\partial^2 u}{\partial \eta^2} + \gamma e^{-\eta} \frac{\partial u}{\partial \tau} + \frac{\partial u}{\partial \eta} - e^{-\eta} w$$
$$K(u, w) = \gamma \frac{\partial u}{\partial \tau} + \frac{\partial w}{\partial \eta} \qquad (1.11)$$

In view of the fact that (1.8) are linear homogeneous equations and their coefficients are independent of the variable τ, we seek the solutions of (1.9) in the form

$$u_1 = \varphi_{01}(\eta) e^{i\tau} + \bar{\varphi}_{01}(\eta) e^{-i\tau}, \qquad w_1 = \psi_{01}(\eta) e^{i\tau} + \bar{\psi}_{01}(\eta) e^{-i\tau} \qquad (1.12)$$

(here the overbar denotes conjugate function).

Substituting (1.12) into (1.8), we obtain for φ_{01} and ψ_{01} the following system of ordinary differential equations

$$\varphi_{01}'' + \varphi_{01}' + i\gamma e^{-\eta} \varphi_{01} - e^{-\eta} \psi_{01} = 0, \quad \psi_{01}' + i\gamma \varphi_{01} = 0 \qquad (1.13)$$

with the boundary conditions

$$\varphi_{01} = \psi_{01} = 0 \quad (\eta = 0), \quad \varphi_{01} \to 1/2 \quad (\eta \to \infty)$$

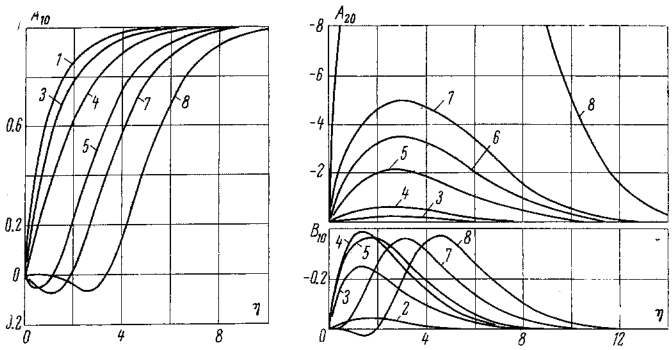

Fig. 1

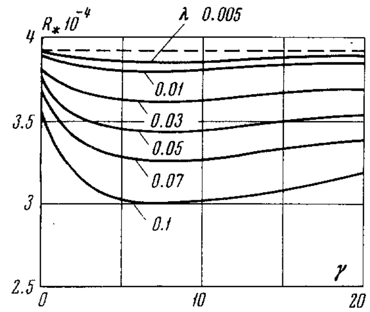

Fig. 2

The solution of (1.13) has the form

$$\varphi_{01}(\eta) = e^{-\eta}\int_0^\eta Q(z)\,e^z\,dz, \qquad \psi_{01}(\eta) = e^{-\eta}\int_0^\eta T(z)\,e^z\,dz \tag{1.14}$$

where

$$Q(z) = e^{-z/2} Z_1(2\sqrt{i\gamma}\,e^{-z/2}), \quad T(z) = Z_0(2\sqrt{i\gamma}\,e^{-z/2})$$

$$Z_i(x) = C_{1i}J_i(x) + C_{2i}Y_i(x)$$

$$C_{11} = \frac{\pi}{2}\sqrt{i\gamma}\,\frac{Y_0(2\sqrt{i\gamma})}{J_0(2\sqrt{i\gamma})}, \quad C_{12} = -\frac{\pi}{2}\sqrt{i\gamma}, \; C_{10} = -\sqrt{i\gamma}\,C_{11}, \; C_{20} = \frac{\pi}{2}i\gamma$$

Here $J_i(x)$ and $Y_i(x)$ are Bessel functions of the first and second kind respectively.

We seek the solutions of (1.9) in the form

$$u_2 = f_2(\eta) + \varphi_{11}(\eta)\,e^{2i\tau} + \overline{\varphi_{11}}(\eta)\,e^{-2i\tau}, \quad w_2 = \psi_{11}(\eta)\,e^{2i\tau} + \overline{\psi}_{11}(\eta)\,e^{-2i\tau} \tag{1.15}$$

Similarly, for the functions u_k, w_k we seek the solutions of (1.10) in the form:

for k = (2n + 1)

$$u_k = \sum_{m=0}^{n} (\varphi_{km}e^{i(2m+1)\tau} + \overline{\varphi}_{km}e^{-i(2m+1)\tau})$$

$$w_k = \sum_{m=0}^{n} (\psi_{km}\,e^{i(2m+1)\tau} + \overline{\psi}_{km}\,e^{-i(2m+1)\tau})$$

for k = 2n

$$u_k = f_k(\eta) + \sum_{m=1}^{n} (\varphi_{km}\,e^{i2m\tau} + \overline{\varphi}_{km}\,e^{-i2m\tau})$$

$$w_k = \sum_{m=1}^{n} (\psi_{km}\,e^{i2m\tau} + \overline{\psi}_{km}\,e^{-i2m\tau})$$

Substituting (1.15) into (1.9), we obtain for f_2, φ_{11}, and ψ_{11} the system of equations

$$f_2'' + f_2' = \varphi_{01}'\,\overline{\psi_{01}} + \overline{\varphi}_{01}'\,\psi_{01} \tag{1.16}$$

$$\varphi_{11}'' + \varphi_{11}' + 2i\gamma\,e^{-\eta}\varphi_{11} - e^{-\eta}\varphi_{11} = -{}^1/_4\,i\gamma + i\gamma\,\varphi_{01}^2 + \varphi_{01}'\psi_{01} \tag{1.17}$$

$$\psi_{11}' + 2\gamma\,\varphi_{11} = 0 \tag{1.18}$$

with the boundary conditions

$$f_2 = \varphi_{11} = \psi_{11} = 0 \quad (\eta = 0), \qquad f_2 \to 0, \quad \varphi_{11} \to 0 \quad (\eta \to \infty)$$

The solution for $f_2(\eta)$ can be written in the form

$$f_2(\eta) = e^{-\eta}\int_0^{\eta} e^t \int_t^{\infty} [\varphi_{01}'(z)\,\bar{\psi}_{01}(z) + \overline{\varphi_{01}'}(z)\,\psi_{01}(z)]\,dz\,dt \tag{1.19}$$

The solutions for φ_{11} and ψ_{11} and also for the remaining φ_{km} and ψ_{km} can be constructed formally after determining the Green's function for the homogeneous system (1.13) with zero boundary conditions. It is not difficult to obtain the Green's function, using the solution (1.14). For the functions $f_k(\eta)$ the solutions are analogous to (1.19). The use of a computer is necessary for concrete construction of the functions f_k, φ_{km}, and ψ_{km}.

As was done in [1], the solutions for u_1, w_1 and u_2, w_2 can be represented in the form

$$\begin{aligned} u_1(\eta,\tau) &= A_{10}(\eta)\cos\tau + B_{10}(\eta)\sin\tau \\ w_1(\eta,\tau) &= A_{10}^*(\eta)\cos\tau + B_{10}^*(\eta)\sin\tau \\ u_2(\eta,\tau) &= A_{20}(\eta) + A_{21}(\eta)\cos 2\tau + B_{21}(\eta)\sin 2\tau \\ w_2(\eta,\tau) &= A_{21}^*(\eta)\cos 2\tau + B_{21}^*(\eta)\sin 2\tau \end{aligned} \tag{1.20}$$

where

$$\begin{gathered} A_{10} = 2\mathrm{Re}\,\varphi_{01}, \quad B_{10} = -2\mathrm{Im}\,\varphi_{01}, \quad A_{10}^* = 2\mathrm{Re}\,\psi_{01} \\ B_{10}^* = -2\mathrm{Im}\,\psi_{01}, \; A_{20} = f_2, \quad A_{21} = 2\,\mathrm{Re}\,\varphi_{11} \\ B_{21} = -2\,\mathrm{Im}\,\varphi_{11}, \quad A_{21}^* = 2\mathrm{Re}\,\psi_{11}, \; B_{21}^* = -2\mathrm{Im}\,\psi_{11} \end{gathered}$$

Figure 1 shows the functions A_{10}, B_{10}, and A_{20}, where the following values of γ correspond to curves 1, . . . , 8.

Curves	1	2	3	4	5	6	7	8
$\gamma =$	0	0.1	0.5	1	3	5	7	30

Let us compare this solution with the solution of [2] for the boundary layer with uniform suction when the external flow has the velocity $U(t) = U_0(1 + \lambda\cos\omega t)$. Just as for boundary layers without suction [1], the behavior of the flow in the boundary layer in these two cases is significantly different. It is interesting to note that for both cases the behavior of the flow in the boundary layer with uniform suction agrees qualitatively with that of the corresponding boundary layer without suction.

2. Study of Flow Stability in the Boundary Layer. The basic flow whose stability is studied is the flow obtained in Section 1 in the boundary layer of a flat plate with uniform suction when the outer flow is given by (1.1). Neglecting, as is usually done in examining boundary layer flow stability, the lengthwise nonuniformity of the stream and the transverse velocity component, the flow in the boundary layer can be considered approximately plane-parallel with the longitudinal velocity $u(y, t)$.

We shall use the very simple quasistationary definition of nonstationary flow stability, i.e., for each moment of time we determine the value of the critical Reynolds number R as for stationary flow and we take as the unknown the minimal value of R in the limits of a single period for the problem being examined.

To study the stability we use the Lin equation [3] for approximate determination of the minimal value of the Reynolds number on the neutral curve, obtained in the small oscillation method of hydrodynamic stability theory.

The Lin equation has the form

$$R \approx \frac{25\,\overline{U}'(0)}{c^4}, \qquad c = \overline{U}(y_c) \tag{2.1}$$

where y_c is the root of the equation

$$-\pi\,\overline{U}'(0)\left\{3 - 2\,\frac{\overline{U}'(0)\,y_c}{\overline{U}(y_c)}\right\}\frac{\overline{U}(y_c)\,\overline{U}''(y_c)}{\overline{U}'^3(y_c)} = 0.58 \tag{2.2}$$

The dimensionless variables are

$$y = \delta\bar{y}, \quad u = U\,\overline{U}, \quad R = U\delta/\nu \tag{2.3}$$

Here δ is the boundary-layer thickness, defined as the distance from the wall to the point where the velocity $u = 0.999\,U$ (prime denotes differentiation with respect to $\bar{y}$).

Figure 2 shows the critical Reynolds number

$$R_* = R\,\frac{U_0\delta^*}{U\delta}$$

based on the displacement thickness $\delta^* = \nu/|v_0|$ and the velocity U_0 versus the parameter γ and the oscillation amplitude λ. The dashed straight line corresponds to $R_* = 3.93 \cdot 10^4$. The Lin equation yields this value of R_* for the stationary asymptotic suction profile (1.7).

Analyzing the influence of the frequency ω on the value of R_*, we see that, just as for the boundary layer without suction [1], there is a most "dangerous" range of frequencies in which R_* takes minimal values.

LITERATURE CITED

1. I. V. Pushkareva, "Laminar boundary layer on flat plate in oscillating flow," PMTF, 9, No. 4 (1968).
2. J. T. Stuart, "A solution of the Navier–Stokes and energy equations illustrating the response of skin friction and temperature of an infinite plate thermometer to fluctuations in the stream velocity," Proc. Roy. Soc., Ser. A, 231, No. 1184 (1955).
3. N. E. Kochin, I. A. Kibel', and N. V. Roze, Theoretical Hydromechanics, Part 2 [in Russian], Fizmatgiz, Moscow (1963).